Instability in a slightly inclined water channel

By A. M. BINNIE
Engineering Laboratory, Cambridge

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An examination was made of the laminar travelling waves that disturbed the glassy surface in an open water channel, fitted with a trumpet entry and inclined downwards over the range 1 to $2\frac{3}{4}°$. The Reynolds number at their first appearance was observed, and measurements were obtained of their velocities and lengths, the latter being highly irregular. Comparison was made with the theoretical values after corrections had been introduced to allow for the higher velocities of flow near the walls due to the increased depth caused by surface tension. At greater discharges the onset of turbulence occurred in a random manner, leading to the production of intermittent bores moving more slowly than the laminar stream. Transition was found to have usually begun at a Reynolds number $R = 2500$, R being defined as the discharge per unit width divided by the kinematic viscosity. Except close to the inlet, the channel was continuously occupied by turbulent water when R was in the range 4000 to 4500.

1. Introduction

Two different kinds of instability can be observed in a long open water channel, fitted with a trumpet entry and inclined at a few degrees below the horizontal. As the supply is slowly increased from zero, the nature of the flow is found to change in the following sequence. As soon as a continuous film is formed covering the whole bottom, the motion is laminar, and the surface is glassy and undisturbed. This surface then becomes unstable, and transverse travelling waves appear, the theory of which has been given by Brooke Benjamin (1957, 1958). These waves increase in size; and when the velocity of the surface exceeds about 9 in./sec, stationary capillary waves spring from irregularities in the wetting of the walls in the manner mentioned by several investigators, e.g. Jeffreys (1925). At once, the travelling waves are apparently suppressed, and the surface again becomes seemingly almost motionless though covered by a diagonal pattern of stationary ripples. The next change is the irregular appearance of spots of turbulence, which immediately give rise to turbulent bores moving down the channel slower than the laminar stream. These spots become more and more numerous and originate nearer the entry, until finally the turbulence extends continuously throughout the entire channel except close to the trumpet.

In the following pages an account is given of these occurrences, as seen in apparatus described in § 2. The theory of the laminar travelling waves is outlined in § 3, and this is followed in § 4 by an account of the experiments on them. Details of the development of turbulence are explained in § 5.

2. Description of apparatus

The channel, which was made for general laboratory purposes, is composed of three sheets of plate glass 16 ft. long. For constructional reasons, each bottom joint consists of a teak strip as shown in the inset to figure 2. The clear distance between the strips is 3·30 in.; and for depths not exceeding $\frac{1}{20}$ in., as in many of the laminar wave experiments, the sides of the stream were vertical. At somewhat greater depths the walls at the free surface were inclined at 45°; but in view of the imprecise nature of the observations, it was thought unnecessary to attempt the awkward depth measurements, from which the slight widening of the stream could be calculated.

The bottom glass plate is fixed to a Dural channel placed web uppermost, a small gap being left between them that permits the insertion of graduated scales, and of white surfaces to assist observations. This channel is supported on a journal bearing close to the supply reservoir and on a jack near the outlet end, which allows the channel to be set at downward inclinations as far as $2\frac{3}{4}$°. A plywood trumpet, protruding into the reservoir, is rigidly attached to the channel, and a joint of flexible material between the reservoir wall and the exterior of the trumpet provides the necessary freedom of movement. The reservoir is 4 × 4 ft. in plan, and it extends about 28 in. below the level of the channel entry. It is fed by a submerged pipe from the laboratory circulating system; and to avoid the settlement of dust, the apparatus was run continuously for weeks on end while the observations were in progress. The water temperature was not under control; and, measured in the reservoir, it altered slowly between the extremes 13 and 19 °C. The discharge was determined by means of graduated glass cylinders and a small tank, into which the water fell after passing through the channel. The few depth measurements made were obtained with a point gauge mounted on a travelling carriage. The apparatus rested on a concrete floor, and the more delicate observations were postponed until no machinery in the building was audible.

3. The theory of wave formation in a laminar stream of infinite width

For reference, a summary of the properties of the undisturbed flow, free from the influence of side walls, will first be given in a form that will be wanted in § 4. The well-known analysis due to Nusselt (1916) shows that, if Q is the discharge per unit width of a stream of liquid of kinematic viscosity ν on a plane inclined at an angle θ to the horizontal, the surface velocity u_0 can be calculated from Q by means of the relation

$$u_0 = \left(\frac{9}{8}\frac{Q^2}{\nu} g \sin\theta\right)^{\frac{1}{3}}. \tag{3.1}$$

Likewise, the depth h of the stream is given by

$$h = \left(\frac{3\nu Q}{g \sin\theta}\right)^{\frac{1}{3}}. \tag{3.2}$$

The Reynolds number R will be defined by

$$R = \frac{Q}{\nu}. \tag{3.3}$$

The stability of this motion was examined in great detail by Brooke Benjamin (1957, 1958), who introduced the dimensionless phase velocity c and wave-number $\alpha = 2\pi h/\text{wavelength}$, and assumed R and α to be fairly small. Supposing that (with u_0 as the unit) $c = c_r + ic_i$, where c_r and c_i are wholly real, he found by means of a simplified theory, valid for long waves of infinitely small amplitude, that

$$c_r = 2, \tag{3.4}$$

$$c_i = \tfrac{1}{2}R\left(\tfrac{8}{5}\alpha - \frac{\alpha^3\Gamma}{hu_0^2} - \frac{\alpha h}{u_0^2}g\cos\theta\right). \tag{3.5}$$

Here Γ is the kinematic surface-tension, defined as the usual surface tension divided by the density. It is seen from (3.5) that the condition for instability is

$$\frac{8}{5} > \frac{\alpha^2\Gamma}{hu_0^2} + \frac{h}{u_0^2}g\cos\theta. \tag{3.6}$$

Following the method that he applied to the case $\theta = 90°$, we will find α for the most unstable wave by maximizing

$$\alpha c_i = \tfrac{1}{2}R\left(\tfrac{8}{5}\alpha^2 - \frac{\alpha^4\Gamma}{hu_0^2} - \frac{\alpha^2 h}{u_0^2}g\cos\theta\right). \tag{3.7}$$

Hence, the optimum value a_m of α is given by

$$\alpha_m^2 = \frac{hu_0^2}{\Gamma}\left(\frac{4}{5} - \frac{h}{2u_0^2}g\cos\theta\right). \tag{3.8}$$

With the aid of the primary-flow equations (3.1) to (3.3) this result can be expressed in terms of the Reynolds number as

$$\alpha_m = \frac{3^{\frac{5}{6}}}{2}\left(\frac{4}{5} - \frac{2}{3}\frac{\cot\theta}{R}\right)^{\frac{1}{2}}\frac{\nu^{\frac{2}{3}}}{\Gamma^{\frac{1}{2}}}(g\sin\theta)^{\frac{1}{6}}R^{\frac{5}{6}}. \tag{3.9}$$

We then have from (3.5) that the maximum value of c_i is

$$(c_i)_m = \frac{3^{\frac{5}{6}}}{4}\left(\frac{4}{5} - \frac{2}{3}\frac{\cot\theta}{R}\right)^{\frac{3}{2}}\frac{\nu^{\frac{2}{3}}}{\Gamma^{\frac{1}{2}}}(g\sin\theta)^{\frac{1}{6}}R^{\frac{11}{6}}. \tag{3.10}$$

Brooke Benjamin calculated the amplification factor

$$\mathscr{A} = \exp\{10\alpha_m(c_i)_m/2h\}, \tag{3.11}$$

where h is in cm, experienced by the wave of maximum instability on a vertical water film at 19 °C as it travels 10 cm. For a plane inclined at an angle θ and with his values $\nu = 0{\cdot}0103\ \text{cm}^2/\text{sec}$, $\Gamma = 72{\cdot}9\ \text{cm}^3/\text{sec}^2$, $g = 981\ \text{cm}/\text{sec}^2$, it appears from (3.2), (3.3), (3.9) and (3.10) that (3.11) becomes

$$\mathscr{A} = \exp\left\{0{\cdot}0434\left(\frac{8}{5} - \frac{4}{3}\frac{\cot\theta}{R}\right)^2(\sin^{\frac{2}{3}}\theta)R^{\frac{7}{3}}\right\}. \tag{3.12}$$

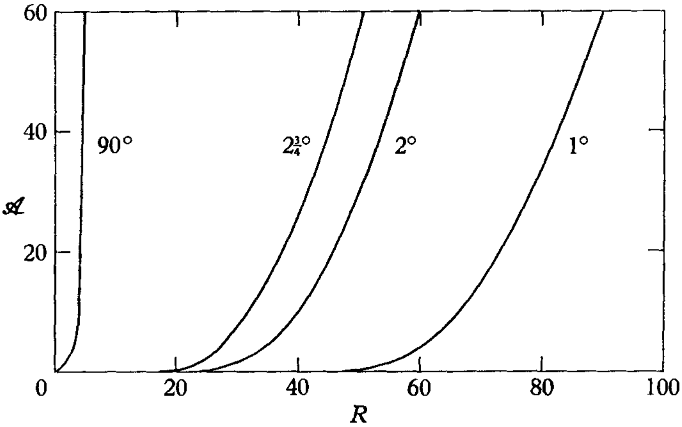

FIGURE 1. The amplification factor $\mathscr{A}$.

For the three values $\theta = 2\frac{3}{4}, 2, 1°$ that were used in the experiments, the relation (3.12) is plotted in figure 1, to which Brooke Benjamin's curve for $\theta = 90°$ has been added. As is clear from (3.5), for $\theta < 90°$ a definite value of $R > 0$ exists below which waves cannot develop, whereas for $\theta = 90°$ the flow is always unstable. The diagram indicates an important tendency. For $\theta = 90°$ the onset of large values of $\mathscr{A}$ is very sudden, and the theory did in fact confirm Binnie's experimental result (1957) that waves were just visible at $R = 4{\cdot}4$. On the other hand, the curves for small values of θ are far from steep; and it is not to be expected that the first appearance of waves as R is increased will be well defined.

4. Experiments on laminar travelling waves in the channel

The object of these experiments was to determine the Reynolds number at which the waves were just visible, together with the velocity and length of these waves. Preliminary observations showed that the waves formed near the inlet and were influenced by the sharp outlet edge where draw-down and surface-tension had some influence. In between, although some died away and fresh ones appeared, the general effect was independent of position in the channel. Measurements were therefore confined to a region about 8 ft. from the inlet, and this was illuminated by a 750 W projector, which produced a beam free from heat and inclined upstream at about 30° to the water surface. At low values of R the waves moved in an orderly manner with none of the obvious overtaking of small waves by large that was seen in the vertical film experiments. As in that investigation the wavelength was irregular, but now the waves were much less steep, and the shadows that they cast on a white surface below the glass bottom were diffuse and not easy to observe.

One effect of the side walls was immediately evident, namely, the waves did not extend right across the channel. The film thickness was too small to be accurately determined with a point gauge, and the matter was investigated by measurements of the surface velocity. Straight hairs about $\frac{1}{4}$ in. long were floated down, and their passage over 22 in. timed with a stop-watch; rotation of the hair was rarely noticed. It was found that over most of the width the surface velocity was almost uniform, but that near the walls a narrow band existed having a higher velocity.

With very thin films this velocity was as much as twice the central velocity. The cause of this effect was surface tension, which increased the depth near the walls and thereby permitted a rise in speed. Therefore, in order that a comparison could be made with the theory of § 3, it was necessary to apply a negative correction to the value of R deduced from the measured discharge. This correction was larger than the theoretical effect of the walls due to viscous action alone, and it lay in the opposite sense. For Cornish's analysis (1928) reveals that, for a stream of viscous liquid devoid of surface tension, the reduction of discharge due to the walls is only $1\frac{1}{4}$ % when the ratio of depth to breadth is 1 %, a value representative of the experiments. We shall use the terms 'net u_0' and 'net R' to refer to the observed central surface velocity and to the Reynolds number calculated from it by means of (3.1) and (3.3), and the terms 'gross u_0' and 'gross R' to refer to the surface velocity and the Reynolds number calculated from the measured discharge by means of the same equations.

Hair measurements of the central velocity were made over the ranges required for the various wave observations, the mean of at least ten timings being taken. At the higher velocities, the uniformity of the hair's travel was slightly disturbed by the waves which had developed, and for the 1° slope the accuracy of the method fell away because the time of passage became rather short. The results are displayed in figures 2 and 3. The former, which gives the ratio of the net to the gross values of u_0, indicates that the difference between the two was large for the thin films existing at the $2\frac{3}{4}$° slope, but became only 10 % at 1° slope. This tendency is in accord with (3.2) and (3.3), which show that at a fixed value of R the stream becomes deeper as the inclination is diminished, therefore the consequencies of surface tension at the walls are proportionately smaller. The latter diagram is of more immediate use in that it provides the net value of R when the gross R has been determined. Again, the 1° results lie above the others, being less disturbed by wall effects.

Three methods were used of discerning the onset of wave formation more sensitively than observation of the shadows (mentioned above) on the white surface beneath the bottom.

A. A vertical screen was set up that intercepted the beam from the projector after reflexion from the water surface. On this, disturbances showed as moving irregularities in the intensity of illumination.

B. Direct but necessarily distant scrutiny was made from the outlet end of the channel at a small incident angle.

C. An electronic capacity indicator was devised by connecting the output from a Feilden Proximity Meter through an amplifier to a cathode-ray oscilloscope, the apparatus being connected to an electrode of diameter 1 in. held just above the water. When a balance had been obtained, a steady curve was visible on the screen. The passage of waves was indicated by momentary disturbances in the curve, the agreement between the frequencies of their occurrence and of waves actually seen in the channel being very convincing. No attempt was made to determine the sensitivity of the arrangement, but it was found that, when the electrode was raised 0·01 in. from its working position, the disturbances became so enormous that their peaks were thrown far off the screen.

An estimate of the first appearance of waves was obtained by increasing and decreasing the discharge until it was thought that the best assessment had been made. Methods A and B gave almost identical results, and the gross values of R produced by the former are shown in table 1. Method C was employed on a later occasion; and, as will be seen in the table, for the $2\frac{3}{4}°$ slope it yielded virtually the same result as A. This suggests that, as the discharge was increased, the development of the waves was rapid. In contrast, at 1° method C detected waves at

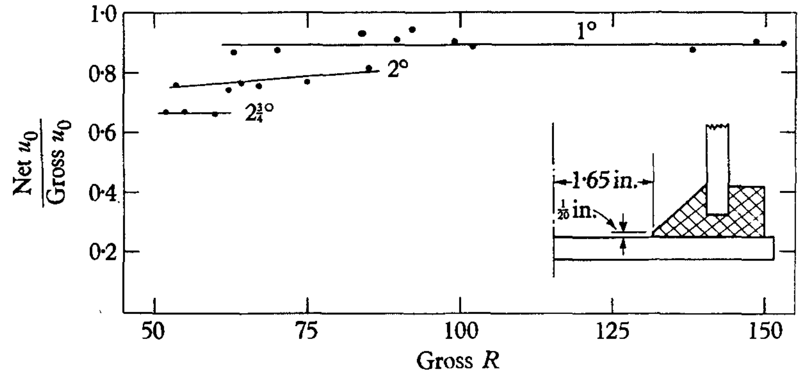

FIGURE 2. The ratio of net to gross u_0. Inset: half cross-section of channel.

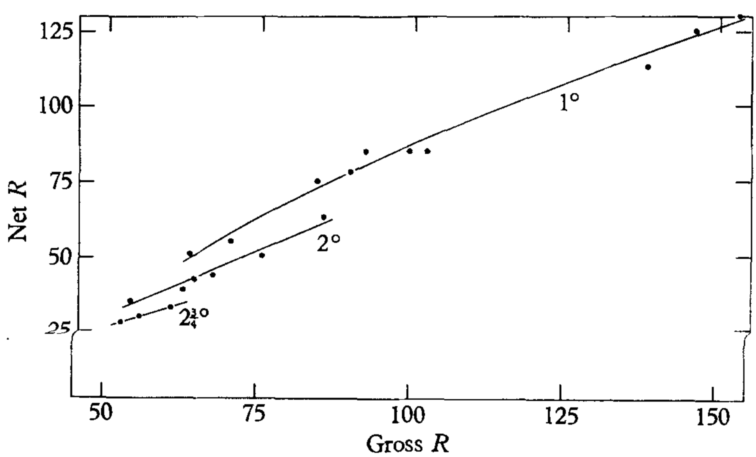

FIGURE 3. The net Reynolds number.

rather smaller discharges than A; thus the wave growth was now slower if it may be assumed that the sensitivity of A remained constant. The table also gives for method C the values of net R obtained from figure 3 and, for comparison, the Reynolds number for zero amplification calculated from the relation $R = \frac{5}{6}\cot\theta$, which follows from (3.11). The differences between these last two columns are roughly the same, the discrepancy at 1° being least, both actually and proportionately.

The disturbances shown on the vertical screen could not be interpreted to yield the velocity and length of the waves; therefore, to determine these quantities, vertical photographs were taken of the shadows thrown on the white surface below the bottom. A 16 mm cine camera was used with a clock and a scale in the field of view. In order to obtain waves big enough to be photographed, it was necessary to raise R above the values given in table 1. As will be seen in table 2, at $2\frac{3}{4}°$ the required increase of gross R was only from 54 to 59, whereas at 1° the waves were so ill defined and developed so slowly that the gross R had to be raised from 69 to as much as 145 before measurable photographs could be secured. Thus the tendency shown up by the differences between the results of methods A and C in table 1 was found again.

The wave velocity was measured by following the passage of individual waves for distances up to 13·2 in., and taking the mean. To compare it with the central surface velocity, the values of net u_0 have been added to table 2, together with the ratios of the observed wave velocity to net u_0. Equation (4.3) indicates that theoretically the ratio is 2; thus there are discrepancies here, which are in the

Slope	Method A, gross R	Method C: Gross R	Method C: Net R	Theoretical minimum R
$2\frac{3}{4}°$	55	54	29	17
2°	64	57	37	24
1°	90	69	56	48

TABLE 1. The onset of wave formation.

Slope	Gross R	Net R	Wave velocity (in./sec)	Net u_0 (in./sec)	Wave velocity ÷ net u_0	Observed wave-length (in.)	Theoretical wave-length (in.)	α_m
$2\frac{3}{4}°$	59	32	6·24	3·30	1·89	1·15	1·42	0·143
2°	79	56	7·37	4·27	1·73	1·15	1·12	0·242
1°	145	123	9·83	5·68	1·73	1·11	1·04	0·429

TABLE 2. Photographic observations and comparison with theory.

opposite direction to that found in Binnie's vertical experiments (1957) where the ratio was 2·34, but in that investigation u_0 was calculated from the discharge and was not measured directly. The best agreement was at $2\frac{3}{4}°$, where the increase of discharge required for satisfactory photographs was least. Whenever a fairly regular train of at least 5 waves was noticed, the length was measured, the greatest number of waves being 7, but the mean values must be treated with reserve because the spacing of the waves was generally erratic. For comparison, the theoretical wavelength has been calculated from the net R with the aid of (3.9), h being found from (3.2) and (3.3). As in the vertical experiments, the agreement is surprisingly good, and it may be added that at $2\frac{3}{4}°$ a train of 3 waves of length 1·45 in. was also observed. The last column in table 2 gives the values of α_m obtained from (3.9) which are included to show that, although R was rather high for the theory to be valid, yet α_m remained small, and the assumption that the waves were long compared with h was not violated. In the vertical experiments α_m was 0·067.

A trial was also made with the channel set at an inclination of only 10′. As the discharge was increased from zero, the first disturbance of the stream was due to the stationary ripples springing from the walls, and later turbulence set in. No laminar travelling waves could be discerned.

5. The development of turbulence

When the supply was raised beyond the range considered in the previous section, the amplitude of the waves increased with little alteration in their distances apart. They were soon obscured by the stationary ripples inclined to the walls at angles which diminished with rising discharge. The first sign of turbulence was an obvious commotion in the surface starting at a fixed point on one wall and spreading diagonally across to the other. This kind of disturbance was clearly the same as the turbulent spots developing into triangular shape, which were seen by Emmons (1951) on an inclined plate 3ft. wide. In the present circumstances, however, the lateral development of the disturbance was limited by the far wall; and blocks of agitated water travelled down the channel each with a normal front and a diagonal tail, their length depending on the duration of the originating outbreak. Evidently they had much in common with the flashes, which were seen by Reynolds (1883) in his classical research on pipe flow, and which have recently been investigated in great detail by Lindgren (1953, 1954, 1957) by means of birefringence and pressure measurements. But, whereas in the pipe experiments the liquid in a flash was forced onward with its mean axial velocity virtually unimpaired, in the channel the disturbance had a depth greater than that of the adjacent laminar stream and its velocity was less. Indeed, sometimes the discharge from the channel momentarily stopped altogether as the turbulent front approached the end. The length of the disturbance increased as it travelled down the channel; and the tail, which remained sharply defined, was fed by the laminar stream behind it under conditions resembling those in an ordinary hydraulic jump. Thus the development of a disturbance differed from that of a pipe flash, for which at the higher Reynolds numbers the velocities of the front and tail were found by Lindgren (1957, figure 5.12) to be respectively greater and less than the mean flow velocity.

Usually, but by no means always, the first disturbances occurred far down the channel. At a higher supply their origin moved nearer the inlet, and they became longer and more numerous. At a fixed supply some of the fronts moved faster than others, and occasionally they formed so violently that they were preceded and later outdistanced by a short group of waves on the laminar stream. In the early stages, two kinds of apparently spurious disturbances were noticed: (i) The supply being increased in small steps, disturbances sometimes occurred soon after the valve was opened; but after a few minutes had elapsed and the supply had become steady, the surface remained unruffled for a long time. It appeared that the flow was in so unstable a state that it was upset by the wetting of surfaces previously dry. (ii) On a few occasions the work was impeded by the deposit of minute air bubbles on the bottom. When seen, they were at once brushed off; nevertheless, a disturbance could occur which carried down in front of it a single large bubble, evidently formed by the sweeping action of the bore. These occurrences were rejected as being possibly caused by the minute bubbles.

Observations with increasing supply were made at slopes of $2\frac{3}{4}$, 2, 1°, and no significant change was noticed on repetition with a decreasing supply. The process of transition was random both in time and place; and if the changes of slope did in fact influence it, their power to do so was obscured by the capricious nature of the disturbances. The observations may be summarized by stating that over a period of minutes at least one disturbance usually appeared when R was 2500. This value, like those given in this section, is a mean, calculated from the discharge, the irregularities in which were short compared with the time of measurement.

At $2\frac{3}{4}$° slope the bores were more conspicuous than at the lesser slopes. The depth of a typical disturbance at 1° slope and $R = 3100$ had a maximum value about 0·28 in. and diminished towards the ends; after a wait for a quiet period of sufficient duration the depth of the laminar stream was found to be 0·18 in. Additional evidence concerning the large difference between the depths in turbulent and laminar flow is provided in figure 4, which shows the results when

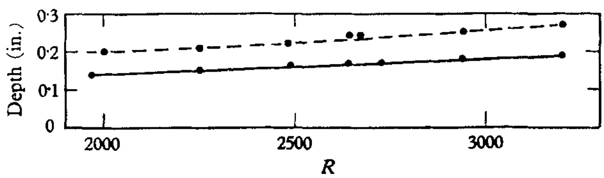

FIGURE 4. The depths at 1° slope: —— laminar, ---- turbulent.

turbulence, probably fully developed, was induced throughout the channel by means of a brass post, $\frac{5}{8}$ in. in diameter, placed on the centre line of the channel at its inlet. The diagram extends as far as $R = 3200$, beyond which the periods of laminar motion were too brief for measurements of its depth to be made. These depths were much larger than those encountered in the laminar wave experiments, and corrections of the kind described in § 4 were not required. The observation 0·28 in., mentioned above, lies close to the turbulent curve, but evidence produced in the next paragraph suggests that in self-induced disturbances at greater values of R the turbulence was not quite fully developed.

The supply being further increased, the disturbances became still more frequent and all began between 1 and 2 ft. from the channel inlet. Their origin extended right across the channel, hence their tails as well as their fronts were normal to the walls. Again, the effect of altering the slope was not clear, and it is impossible to give precise numerical results, though at $R = 3500$ there was almost always at least one disturbance in the channel and at $R = 4000$ to 4500 the disturbances were generated continuously. But even in this range the discharge was made slightly irregular by the occasional appearance of roll waves in the downstream half of the channel. At 1° slope and $R = 4100$, these waves vanished when the turbulence stimulator was placed at the channel inlet, the depth increasing from 0·33 in. by roughly 0·01 in. This observation is in agreement with Rouse's remark (1938) that roll waves can be suppressed by roughening the walls, although at first sight it is surprising that so small an increase in the apparent turbulence was effective. However, on the assumption that the Chézy resistance law holds good, the criterion for the existence of roll waves put forward by Jeffreys (1925) and

Dressler (1952) is $\tan\theta > 4r^2$, the friction force per unit mass being r^2 multiplied by the square of the mean velocity and divided by the depth. From the measurements of depth and discharge without turbulence stimulation, $4r^2$ was estimated to be 0·018, which is not far from $\tan 1^\circ = 0{\cdot}0175$. When R was raised to 5500, roll waves were absent. Thus the transition to turbulence was an ill-defined process, and the numerical results obtained might well have been different had the channel been supplied from a larger reservoir through a contraction of still better design.

The surface near the inlet which was never troubled by turbulence could be examined by watching the reflexion of a roof light through the network of stationary ripples. Even when R was less than 2000 the surface quivered as a commotion boiled up from below and was quickly damped out. At higher values of R, the agitation was more marked; nevertheless, some conspicuous heaves in the surface faded away after travelling a short distance, their suppression probably being influenced by surface tension as well as by gravity and viscosity. When attention was concentrated on a point a foot or two downstream from the inlet, sometimes a heave in the surface was seen to be succeeded by the appearance of turbulence. Presumably the surface turbulence was caused, not by the passage of the water through the stationary ripples, but by eddies diffused from the bottom. At the highest Reynolds numbers the surface in the channel, that was free from turbulence but shaken continuously in an irregular manner, extended up to the inlet itself, and it could be seen that eddies were generated in the contraction also.

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